Chapter 12

Review on Complex Analysis I

Reading: Alfors [1]:

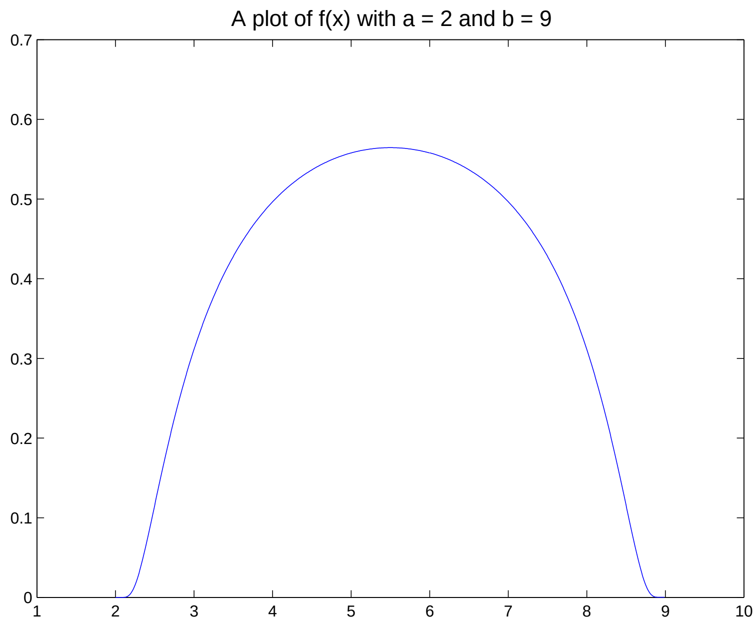
- Chapter 2, 2.4, 3.1-3.4
- Chapter 3, 2.2, 2.3
- Chapter 4, 3.2
- Chapter 5, 1.2

12.1 Cutoff Function

Last time we talked about cutoff function. Here is the way to construct one on $\mathbb{R}^n$ [5].

Proposition 12.1.1. *Let A and B be two disjoint subsets in $\mathbb{R}^m$, A compact and B closed. There exists a differentiable function φ which is identically 1 on A and identically 0 on B*

Proof. We will complete the proof by constructing such a function.



Let $0 < a < b$ and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \left\{ \begin{array}{cc} \exp\left(\frac{1}{x-b} - \frac{1}{x-a}\right) & \text{if } a < x < b \\ 0 & \text{otherwise.} \end{array} \right\} \tag{12.1}$$

It is easy to check that f and the function

$$F(x) = \frac{\int_x^b f(t)\ dt}{\int_a^b f(t)\ dt} \tag{12.2}$$

are differentiable. Note that the function F has value 1 for $x \leq a$ and 0 for $x \geq b$. Thus, the function

$$\psi(x_1, \ldots, x_m) = F(x_1^2 + \ldots + x_m^2) \tag{12.3}$$

is differentiable and has values 1 for $x_1^2 + \ldots + x_m^2 \leq a$ and 0 for $x_1^2 + \ldots + x_m^2 \geq b$.

Let S and S' be two concentric spheres in $\mathbb{R}^m$, $S' \subset S$. By using ψ and

linear transformation, we can construct a differentiable function that has value 1 in the interior of S' and value 0 outside S.

Now, since A is compact, we can find finitely many spheres $S_i (1 \leq i \leq n)$ and the corresponding open balls V_i such that

$$A \subset \bigcup_{i=1}^{n} V_i \tag{12.4}$$

and such that the closed balls $\bar{V}_i$ do not intersect B. $\square$

We can shrink each S_i to a concentric sphere S_i' such that the corresponding open balls V_i' still form a covering of A. Let ψ_i be a differentiable function on $\mathbb{R}^m$ which is identically 1 on B_i' and identically 0 in the complement of V_i', then the function

$$\varphi = 1 - (1 - \psi_1)(1 - \psi_2)\dots(1 - \psi_n) \tag{12.5}$$

is the desired cutoff function.

12.2 Power Series in Complex Plane

In this notes, z and a_i's are complex numbers, $i \in \mathbb{Z}$.

Definition 12.2.1. *Any series in the form*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots + a_n(z - z_0)^n + \dots \tag{12.6}$$

is called power series.

Without loss of generality, we can take z_0 to be 0.

Theorem 12.2.2. *For every power series 12.6 there exists a number R, $0 \leq R \leq \infty$, called the* radius of convergence*, with the following properties:*

1. *The series converges absolutely for every z with $|z| < R$. If $0 \leq \rho \leq R$ the convergence is uniform for $|z| \leq \rho$.*

2. *If $|z| > R$ the terms of the series are unbounded, and the series is consequently divergent.*

3. *In $|z| < R$ the sum of the series is an analytic function. The derivative can be obtained by termwise differentiation, and the derived series has the same radius of convergence.*

Proof. The assertions in the theorem is true if we choose R according to the *Hadamard's formula*

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}. \tag{12.7}$$

The proof of the above formula, along with assertion (1) and (2), can be found in page 39 of Alfors.

For assertion (3), first I will prove that the derived series $\sum_1^\infty n a_n z^{n-1}$ has the same radius of convergence. It suffices to show that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1 \tag{12.8}$$

Let $\sqrt[n]{n} = 1 + \delta_n$. We want to show that $\lim_{n\to\infty} \delta_n = 0$. By the binomial theorem,

$$n = (1+\delta_n)^n > 1 + \frac{1}{2} n(n-1)\delta_n^2 \tag{12.9}$$

which gives

$$\delta_n^2 < \frac{2}{n} \tag{12.10}$$

and thus

$$\lim_{n\to\infty} \delta_n = 0 \tag{12.11}$$

Let us write

$$f(z) = \sum_0^\infty a_n z^n = s_n(z) + R_n(z) \tag{12.12}$$

where

$$s_n(z) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} \tag{12.13}$$

is the partial sum of the series, and

$$R_n(z) = \sum_{k=n}^{\infty} a_k z^k \tag{12.14}$$

is the remainder of the series. Also let

$$f_1(z) = \sum_1^{\infty} n a_n z^{n-1} = \lim_{n\to\infty} s_n'(z). \tag{12.15}$$

If we can show that $f'(z) = f_1(z)$, then we can prove that the sum of the series is an analytic function, and the derivative can be obtained by termwise differentiation.

Consider the identity

$$\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0) = \left(\frac{s_n(z)-s_n(z_0)}{z-z_0} - s_n'(z_0)\right) + (s_n'(z_0) - f_1(z_0)) + \left(\frac{R_n(z)-R_n(z_0)}{z-z_0}\right) \tag{12.16}$$

and assume $z \neq z_0$ and $|z|, |z_0| < \rho < R$. The last term can be rewritten as

$$\sum_{k=n}^{\infty} a_k (z^{k-1} + z^{k-2} z_0 + \ldots + z z_o^{k-2}) = z_0^{k-1}, \tag{12.17}$$

and thus

$$\left|\frac{R_n(z)-R_n(z_0)}{z-z_0}\right| \leq \sum_{k=n}^{\infty} k|a_k|\rho^{k-1} \tag{12.18}$$

Since the left hand side of the inequality is a convergent sequence, we can find n_0 such that for $n \geq n_0$,

$$\left|\frac{R_n(z)-R_n(z_0)}{z-z_0}\right| < \frac{\epsilon}{3}. \tag{12.19}$$

From Eq. 12.15, we know that there is also an n_1 such that for $n \geq n_1$,

$$|s'_n(z_0) - f_1(z_0)| < \frac{\epsilon}{3}. \tag{12.20}$$

Now if we choose $n \geq n_0, n_1$, from the definition of derivative we can find $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies

$$\left| \frac{s_n(z) - s_n(z_0)}{z - z_0} - s'_n(z_0) \right| < \frac{\epsilon}{3}. \tag{12.21}$$

Combining Eq. 12.19, 12.20 and 12.21, we have for $0 < |z - z_0| < \delta$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) \right| < \epsilon \tag{12.22}$$

Thus, we have proved that $f'(z_0)$ exists and equals $f_1(z_0)$. □

12.3 Taylor Series

Note that we have proved that a power series with positive radius of convergence has derivatives of all orders. Explicitly,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots \tag{12.23}$$

$$f'(z) = z_1 + 2a_2 z + 3a_3 z^2 + \dots \tag{12.24}$$

$$f''(z) = 2a_2 + 6a_3 z + 12a_4 z^2 + \dots \tag{12.25}$$

$$\vdots \tag{12.26}$$

$$f^{(k)}(z) = k!a_k + \frac{(k+1)!}{1!} a_{k+1} z + \frac{(k+2)!}{2!} a_{k+2} z^2 + \dots \tag{12.27}$$

Since $a_k = f^{(k)}(0)/k!$, we have the Taylor-Maclaurin series:

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!} z^2 + \dots + \frac{f^{(n)}(0)}{n!} z^n + \dots \tag{12.28}$$

Note that we have proved this only under the assumption that $f(z)$ has a power series development. We did not prove that every analytic function has a Taylor development, but this is what we are going to state without proof. The proof can be found in Chapter 4, Sec. 3.1 of [1].

Theorem 12.3.1. *If $f(z)$ is analytic in the region Ω, containing z_0, then the representation*

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \ldots \tag{12.29}$$

is valid in the largest open disk of center z_0 contained in Ω.

12.3.1 The Exponential Functions

We define the *exponential function* as the solution to the following differential equation:

$$f'(z) = f(z) \tag{12.30}$$

with the initial value $f(0) = 1$. The solution is denoted by e^z and is given by

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots \tag{12.31}$$

Since $R = \limsup_{n\to\infty} \sqrt[n]{n!}$, we can prove that the above series converges if

$$\lim_{n\to\infty} \sqrt[n]{n!} = \infty \tag{12.32}$$

Proposition 12.3.2.

$$e^{a+b} = e^a e^b \tag{12.33}$$

Proof. Since $D(e^z \cdot e^{c-z}) = e^z \cdot e^{c-z} + e^z \cdot (-e^{c-z}) = 0$, we know that $e^z \cdot e^{c-z}$ is a constant. The value can be found by putting $z = 0$, and thus $e^z \cdot e^{c-z} = e^c$. Putting $z = a$ and $c = a + b$, we have the desired result. $\square$

Corollary 12.3.3. $e^z \cdot e^{-z} = 1$*, and thus e^z is never 0.*

Moreover, if $z = x + iy$, we have

$$|e^{iy}|^2 = e^{iy}e^{-iy} = 1 \tag{12.34}$$

and

$$|e^{x+iy}| = |e^x|. \tag{12.35}$$

12.3.2 The Trigonometric Functions

We define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{12.36}$$

In other words,

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots, \tag{12.37}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots. \tag{12.38}$$

From Eq. 12.36, we can obtain the Euler's equation,

$$e^{iz} = \cos z + i \sin z \tag{12.39}$$

and

$$\cos^2 z + \sin^2 z = 1 \tag{12.40}$$

From Eq. 12.39, we can directly find

$$D \cos z = -\sin z, \quad D \sin z = \cos z \tag{12.41}$$

and the additions formulas

$$\cos(a + b) = \cos a \cos b - \sin a \sin b \tag{12.42}$$

$$\sin(a + b) = \sin a \cos b + \cos a \sin b \tag{12.43}$$

12.3.3 The Logarithm

The logarithm function is the inverse of the exponential function. Therefore, $z = \log w$ is a root of the equation $e^z = w$. Since e^z is never 0, we know that the number 0 has no logarithm. For $w \neq 0$ the equation $e^{x+iy} = w$ is equivalent to

$$e^x = |w|, \quad e^{iy} = \frac{w}{|w|} \tag{12.44}$$

The first equation has a unique solution $x = log|w|$, the real logarithm of $|w|$. The second equation is a complex number of absolute value 1. Therefore, one of the solution is in the interval $0 \leq y < 2\pi$. Also, all y that differ from this solution by an integral multiple of 2π satisfy the equation. Therefore, every complex number other that 0 has infinitely many logarithms which differ from each other by multiples of $2\pi i$.

If we denote $\arg w$ to be the imaginary part of $\log w$, then it is interpreted as the angle, measured in radians, between the positive real axis and the half line from 0 through the point w. And thus we can write

$$\log w = \log |w| + i \arg w \tag{12.45}$$

The addition formulas of the exponential function implies that

$$\log(z_1 z_2) = \log z_1 + \log z_2 \tag{12.46}$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{12.47}$$

12.4 Analytic Functions in Regions

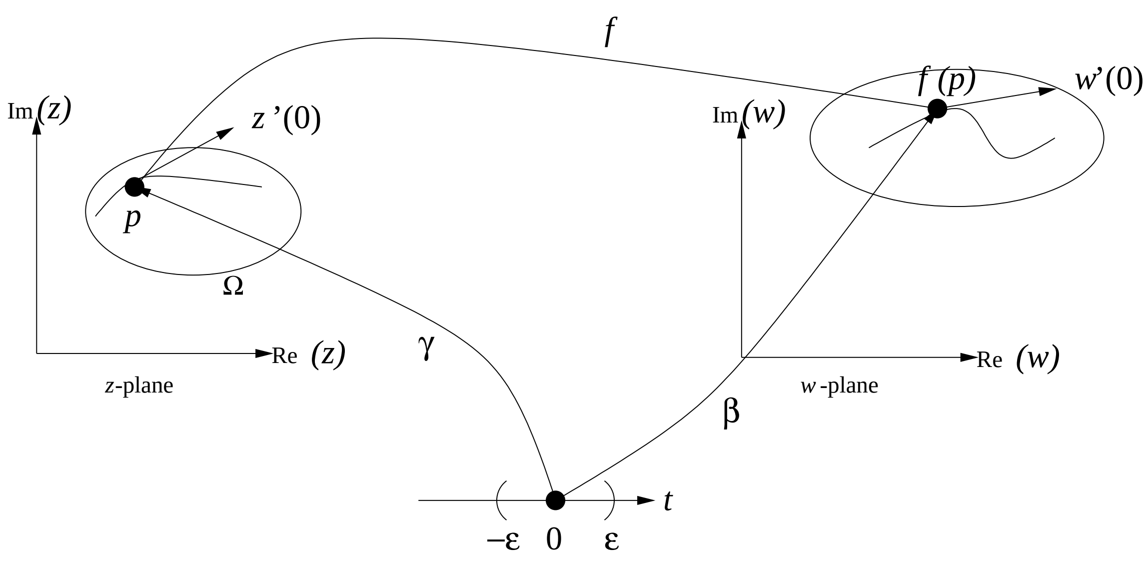
Definition 12.4.1. *A function $f(z)$ is analytic on an arbitrary point set A if it is the restriction to A of a function which is analytic in some open set containing A.*

Although the definition of analytic functions requires them to be single-

valued, we can choose a definite region such that a multiple-valued function, such as $z^{1/2}$, is single-valued and analytic when restricted to the region. For example, for the function $f(z) = z^{1/2}$, we can choose the region Ω to be the complement of the negative real axis. With this choice of Ω, one and only one of the values of $z^{1/2}$ has a positive real part, and thus $f(z)$ is a single-valued function in Ω. The proof of continuity and differentiability of $f(z)$ is in [1] and thus omitted.

12.5 Conformal Mapping

Let γ be an arc with equation $z = z(t), t \in [-\epsilon, \epsilon]$ contained in region Ω with $z(0) = p$. Let $f(z)$ be a continuous function on Ω. The equation $w = w(t) = f(z(t))$ defines an arc β in the w-plane which we call it the image of γ.



We can find $w'(0)$ by

$$w'(0) = f'(p)z'(0). \tag{12.48}$$

The above equation implies that

$$\arg w'(0) = \arg f'(p) + \arg z'(0). \tag{12.49}$$

In words, it means that the angle between the directed tangents to γ at p and to β and $f(p)$ is equal to $\arg f'(p)$, and thus independent of γ. Consequently, curves through p which are tangent to each other are mapped onto curves with a common tangent at $f(p)$. Moreover, two curves which form an angle at p are mapped upon curves forming the same angle. In view of this, we call the mapping f to be *conformal* at all points with $f'(z) \neq 0$.

12.6 Zeros of Analytic Function

The goal of this section is to show that the zeros of analytic functions are isolated.

Proposition 12.6.1. *If f is an analytic function on a region Ω and it does not vanish identically in Ω, then the zeros of f are isolated.*

Proof. Remember that we have assumed in 12.3 that every function f that is analytic in the region Ω can be written as

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \ldots \tag{12.50}$$

Let E_1 be the set on which $f(z)$ and all derivatives vanish at $z_0 \in \mathbb{C}$ and E_2 the set on which the function or one of the derivatives evaluated at z_0 is different from zero. When $f(z)$ and all derivatives vanish at z_0, then $f(z) = 0$ inside the whole region Ω. Thus, E_1 is open. E_2 is open because the function and all derivatives are continuous. Since Ω is connected, we know that either E_1 or E_2 has to be empty. If E_2 is empty, then the function is identically zero. If E_1 is empty, $f(z)$ can never vanish together with all its derivatives.

Assume now that $f(z)$ is not identically zero, and $f(z_0) = 0$. Then there exists a first derivative $f^{(h)}(z_0)$ that is not zero. We say that a is a zero of f of order h. Moreover, it is possible to write

$$f(z) = (z - z_0)^h f_h(z) \tag{12.51}$$

where $f_h(z)$ is analytic and $f_h(z_0) \neq 0$.

Since $f_h(z)$ is continuous, $f_h(z) \neq 0$ in the neighbourhood of z_0 and $z = z_0$ is the unique zero of $f(z)$ in the neighborhood of z_0. $\square$

Corollary 12.6.2. *If f(z) and g(z) are analytic in Ω, and if $f(z) = g(z)$ on a set which has an accumulation point in Ω, then $f(z)$ is identically equal to $g(z)$.*

Proof. Consider the difference $f(z) - g(z)$ and the result from Proposition 12.6.1. $\square$